DIFFERENTIAL GEOMETRY

1. Manifolds and smooth maps

Lem 1.7: f is smooth wrt $\mathcal{A}$ iff $\exists\, p \in X$, $\exists\, \varphi_\alpha$ about p s.t $f \circ \varphi_\alpha^{-1}$ is smooth.

Exm 1.16: $U_\pm := S^n \setminus \{(0,\dots,\pm 1)\}$

$\varphi_\pm := \frac{1}{1 \mp y_{n+1}}(y_1,\dots,y_n)$ local coords: $x_i^\pm = \frac{y_i}{1 \mp y_{n+1}}$

Dfn 1.32: $\gamma_1, \gamma_2 : [0,1] \to X$ agree to first order at p if $\exists$ chart $\varphi : U \to V$ about p s.t. $(\varphi \circ \gamma_1)^\cdot(0) = (\varphi \circ \gamma_2)^\cdot(0)$ $(\gamma_1(0) = \gamma_2(0) = p)$

Dfn 1.35: Tangent space: $T_pX := \{\text{curves based at } p\} / \text{agreement to first order}$

Dfn 1.37: $\frac{\partial}{\partial x_i} := (\pi_p{}^\varphi)^{-1}(e)$; $\pi_p{}^\varphi : \{\text{curves at } p\} \to (\varphi \circ \gamma)^\cdot(0)$.

Lemma 1.38: $\frac{\partial}{\partial y_i} = \sum_j \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}$

Dfn 1.40: $D_pF : T_pX \to T_{F(p)}Y$; $[\gamma] \mapsto [F \circ \gamma]$

Prop 1.43: (Chain rule) $D_p(G \circ F) = D_{F(p)}G \circ D_pF$

2. Vector Bundles and Tensors

Dfn 2.4: a rank - k vector bundle over B is a manifold E along with

- a smooth surjection $\pi : E \to B$, the projection map
- Open cover $\{U_\alpha\}_{\alpha \in A}$ of B s.t $\forall\, \alpha$ $\exists$ diffeomorphism

  $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$

  called a local trivialisation that satisfies:

  1) $pr_1 \circ \Phi_\alpha = \pi$ on $\pi^{-1}(U_\alpha)$

  2) $\forall\, \alpha, \beta$, $\Phi_\beta \circ \Phi_\alpha^{-1}$ is of the form

  $\Phi_\beta \circ \Phi_\alpha^{-1} : (p, v) \mapsto (p, g_{\beta\alpha}(p)v)$

  for some smooth $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$

Exm 2.8: $\underline{\mathbb{R}}^k$ denotes trivial rank - k v.b. over some implicit space.

Dfn 2.10: Section is a smooth map $s : U \to E$; $\pi \circ s = id_B$

Dfn 2.13: $F : B_1 \to B_2$, then a morphism of v.b. is a smooth map $G : E_1 \to E_2$ s.t the diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "G"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "F"] & B_2 \end{tikzcd}$$

and $G : (E_1)_p \to (E_2)_{F(p)}$ is a linear map.

Exm 2.15: a morphism $G : \underline{\mathbb{R}} \to E$ covering identity on B is the same as a global section: $G \rightsquigarrow s : p \mapsto G(p, 1)$, $s \rightsquigarrow G : (p, t) \mapsto t \cdot s(p)$

Motto: E locally trivial iff $\exists$ collection of local sections forming a fibrewise basis.

Exm 2.17: (tautological line bundle over $\mathbb{RP}^n$)

$E = \{(p, v) : v \text{ lies in line } p\}$, $\pi : E \to B$; $(p, v) \mapsto p$

$\{U_0, \dots, U_n\}$, $U_i := \{[x_0, \dots, x_n] : x_i \neq 0\}$

$\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}$; $([x_0 : \dots : x_n], \lambda(x_0, \dots, x_n)) \mapsto ([x_0 : \dots : x_n], \lambda x_i)$

$\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times \mathbb{R} \to (U_i \cap U_j) \times \mathbb{R}$; $([x_0 : \dots : x_n], t) \mapsto ([x_0 : \dots : x_n], \frac{t x_j}{x_i})$

$g_{ji} = \frac{x_j}{x_i} \in \mathbb{R}^*$

Transition functions and cocycle conditions: $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$

- $\forall\, \alpha$, $g_{\alpha\alpha}(p) = I$
- $\forall\, \alpha, \beta, \gamma$, $g_{\beta\alpha}(p) = (g_{\alpha\beta}(p))^{-1}$ and $g_{\alpha\gamma}(p)\, g_{\gamma\beta}(p)\, g_{\beta\alpha}(p) = I$

Exm 2.18: $r \in \mathbb{Z}$, set $g_{ji} = (\frac{x_j}{x_i})^{-r}$. Denote bundle by $\mathcal{O}_{\mathbb{RP}^n}(r)$. Then for $r = -1$, this is the tautological bundle.

Lem 2.19: if $\pi : E \to B$ is a rank k v.b, then it is completely determined by its trivialisation cover $\{U_\alpha\}$ and transition functions $\{g_{\beta\alpha}\}$

Dfn 2.21: $\pi : E \to B$, the pullback bundle via $F : B' \to B$, F^*E, has total space $F^*E := \bigsqcup_{p \in B'} E_{F(p)}$ w/ cover $\{F^{-1}(U_\alpha)\}$, transition functions $g_{\beta\alpha} \circ F$.

Constructing vector bundles by gluing:

Dual bundle: $E^\vee$: fibres $(E_p)^\vee$, $g^\vee_{\beta\alpha} = (g_{\beta\alpha}{}^\vee)^{-1} = (g_{\beta\alpha}{}^T)^{-1}$

Whitney sum: $E_1^k \oplus E_2^\ell$: fibres $(E_1)_p \oplus (E_2)_p$, $g^\oplus_{\beta\alpha} = g^1_{\beta\alpha} \oplus g^2_{\beta\alpha} \in GL(k+\ell, \mathbb{R})$
$\begin{pmatrix} g^1_{\beta\alpha} & 0 \\ 0 & g^2_{\beta\alpha} \end{pmatrix}$

Tensor: $E_1 \otimes E_2$: fibres $(E_1)_p \otimes (E_2)_p$, $g^\otimes_{\beta\alpha} = g^1_{\beta\alpha} \otimes g^2_{\beta\alpha} \in GL(k\ell, \mathbb{R})$

Hom: $Hom(E_1, E_2)$: fibres $Hom((E_1)_p, (E_2)_p)$, $g^{Hom}_{\beta\alpha} := \varphi(g^1_{\beta\alpha}(p), g^2_{\beta\alpha}(p))$

$\varphi : (A, B) \mapsto (X \mapsto B X A^{-1})$

i.e. $g^{Hom}_{\beta\alpha} : Mat_{k \times \ell}(\mathbb{R}) \to GL(k\ell, \mathbb{R})$

Dfn 2.24: Cotangent bundle $= T_p^*X := (T_pX)^\vee$

$\cong \{\text{functions at } p\} / \sim$

where $f_1 \sim f_2$ 'agree up to first order at p' if $D_pf_1 = D_pf_2$

$(\cdot / z_p$, where z_p = vanishing derivative)

pairing $T_p^*X, T_pX : ([f], [\gamma]) \mapsto (f \circ \gamma)^\cdot(0)$

↳ think about $[\gamma] \in T_pX$ as directional derivatives

↳ think about equivalence

$\theta : \underset{T_p^*X}{\underset{\cap}{f}} \longmapsto (\underset{(T_pX)^\vee}{\underset{\cap}{\sum a_i \partial x_i \longmapsto \sum a_i \frac{\partial f}{\partial x_i}\big|_p}})$

Denote $dx^i := \theta(x^i)$ = duals of ∂_{x_i}. $dx^i(\partial_{x_j}) = \delta^i_j$

Lem 2.26: $f: U \to \mathbb{R}$, then df is a smooth section of T^*X given by

$$df = \sum_i \frac{\partial f}{\partial x_i} dx^i$$

Dfn 2.29: Given a smooth map $F: X \to Y$, $(D_p F)^\vee : T_{F(p)}^* Y \to T_p^* X$ is called the pullback of F, denoted F^*

Lem 2.30: $F: X \to Y$, $g: Y \to \mathbb{R}$, then $F^*(dg) = d(g \circ F)$.

3. Differential Forms

Wedge $\Lambda V = {}^{TV}/_{v \otimes v}$, e.g. $v \wedge w = v \otimes w - w \otimes v$

graded commutative: $a \wedge b = (-1)^{|v||w|} b \wedge a$

$\Lambda^n V$ one dimensional, $\alpha: V \to V$ induces map $\det(\alpha): \Lambda^n V \to \Lambda^n V$.
↳ $n = \dim V$

Dfn 3.1: exterior derivative of $\alpha = \alpha_i dx^i$; $d\alpha := \frac{\partial \alpha_i}{\partial x_j} dx^j \wedge dx^i$.

Dfn 3.2: r-form: $\alpha = \alpha_I dx^I$, $d\alpha := \frac{\partial \alpha_i}{\partial x_j} dx^j \wedge dx^I$

Prop 3.4: (i) $d^2 = 0$

(2) $\alpha \in \Omega^r$, $\beta \in \Omega^e$,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

(3) $d(F^*\alpha) = F^*(d\alpha)$ $F: X \to Y$ smooth, $\alpha \in \Omega^r(Y)$

$Z^r(X) = \{$closed r-forms$\}$, $B^r(X) = \{$exact r-forms$\}$

Dfn 3.5: r^{th} de Rham cohomology group: $H_{dR}^r(X) = {Z^r(X)}/{B^r(X)}$

Exm 3.6: $H_{dR}^0(X) = \mathbb{R}^{\#\{\text{connected comp. of } X\}}$

Lem 3.9 (Contravariant functoriality)

$F: X \to Y$ smooth, then $F^*: \Omega^r(Y) \to \Omega^r(X)$ induces a map

$$F^*: H^r_{dR}(Y) \to H^r_{dR}(X)$$

Prop. 3.11: $F_0, F_1: X \to Y$ smoothly homotopic, then they induce the same map $H_{dR}^\cdot(Y) \to H_{dR}^\cdot(X)$.

Cor 3.12: $F: X \to Y$ homotopy equiv, then $F^*: H_{dR}^\cdot(Y) \to H_{dR}^\cdot(X)$ an iso.

Dfn 3.14: an orientation of n-dim. v.s. V is a nonzero element of $\Lambda^n V$, modulo positive rescalings. An ordered basis induces an orientation $e_1 \wedge \cdots \wedge e_n$.

An orientation of a v.b. E is a nowhere-zero section of $\Lambda^{top} E$ modulo rescaling by positive smooth functions. (i.e. $\Lambda^{top} E$ is trivial)

transition function of $\Lambda^{top} E = \det(g_{\beta\alpha})$.

Dfn 3.16: X is oriented if TX is oriented $\Leftrightarrow$ T^*X is oriented.

Dfn 3.18: nowhere vanishing n-form = volume form

Dfn 3.19: $\{U_\alpha\}$ an open cover of X, a partition of unity subordinate to $\{U_\alpha\}$ is a collection of smooth functions $\rho_\alpha : X \to [0,1]$ s.t.

- $\forall \alpha$, $\operatorname{supp}(\rho_\alpha) \subseteq U_\alpha$
- $\forall p \in X$, $\exists$ open nhood U of p s.t all but finitely many ρ_α vanish on U.
- $\sum_\alpha \rho_\alpha = 1$

Dfn 3.21: The integral of ω over X, denoted $\int_X \omega$, is defined as follows

- Cover X by coord patches $\{U_\alpha\}$ so that wlog local coordinates are positively oriented ($\partial x_1 \wedge \cdots \wedge \partial x_n$ coincides w/ orientation on X)
- pick a partition of unity $\{\rho_\alpha\}$ subord. to this cover. Each $\rho_\alpha \omega$ has compact support in U_α. Write in coords as $(\rho_\alpha \omega)_{12\cdots n} dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$
- Define $\int_X \omega = \sum_{\alpha \in A} \int_{\mathbb{R}^n} (\varphi_\alpha \omega)_{12\cdots n} dx^1 \cdots dx^n$.

Theorem 3.27 (Stoke's Theorem)

If X is an oriented MWB and ω compactly supported $(n-1)$-form on X,

$$\int_{\partial X} \iota^* \omega = \int_X d\omega \qquad (\iota: \partial X \to X \text{ inclusion})$$

Aside ∂X is oriented as follows: $p \in X$, T_pX oriented by $O_X \in \Lambda^n T_p X$

Let $\eta \in T_pX$ be any outward pointing normal vector. Then we orient $T_p \partial X$ by $O_{\partial X}$ defined by $O_X = \eta \wedge O_{\partial X}$

Exm 3.28: on $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, oriented by $\partial x^1 \wedge \cdots \wedge \partial x^n$, the vector $-\partial x^1$ is outward pointing so induced orientation on $\{0\} \times \mathbb{R}^{n-1}$ is $-\partial x^2 \wedge \cdots \wedge \partial x^n$

Cor 3.29: $\alpha \in \Omega^{p-1}$, $\beta \in \Omega^{n-p}$, and at least one compactly supported. Then

$$\int_X d\alpha \wedge \beta = \int_{\partial X} \alpha \wedge \beta + (-1)^p \int_X \alpha \wedge d\beta$$

Prop 3.30: X compact, then $\int_X : \Omega^n(X) \to \mathbb{R}$ induces a map
no boundary
$H_{dR}^n(X) \to \mathbb{R}$

4. Flows and Derivatives

flow along vector field v: idea: $p \in X$, find a curve $\gamma: [0,1] \to X$ s.t $\dot\gamma(t) = v(\gamma(t))$, $\gamma(0) = p$

ODE theory $\Rightarrow$ $\exists!$ solution on some $(-\varepsilon, \varepsilon)$, depending smoothly on p

↳ define γ to be an 'integral curve'

Dfn 4.2: a local flow of V comprises a flow domain U and a smooth map $\Phi: U \subset \mathbb{R} \times X \to X$ such that

- $\Phi(0,p) = p$
- $\frac{d}{dt}\Big|\Phi(t,p)\Big) = v(\Phi(t,p))$ $\qquad \Phi^t := \Phi(t,-)$

1-parameter family (in p) of integral curves.

Prop 4.3: $\Phi^{t+s} = \Phi^t \circ \Phi^s$

Dfn 4.4: a vector field is complete if it admits a global flow.

not all global, but all compactly supported are global.

Dfn 4.5: The Lie derivative of a tensor T on X along v is

$$\mathcal{L}_v T := \frac{d}{dt}\Big|_{t=0} (\Phi^t)^* T$$

Rem: $\frac{d}{dt}(\Phi^t)^* T = \frac{d}{dh}\Big|_{h=0} (\Phi^{t+h})^* T = (\Phi^t)^* \mathcal{L}_v T$

Lem 4.6: $f \in C^\infty(M)$, $\mathcal{L}_v f = df(v)$

$$\omega \in \mathfrak{X}(M), \quad \mathcal{L}_v \omega = \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

Lem 4.7: 1-form S and vector field T,

$$\mathcal{L}_v (S_a T^a) = (\mathcal{L}_v S_a) T^a + S_a (\mathcal{L}_v T^a)$$

any tensors S and T,

$$\mathcal{L}_v (S \otimes T) = \mathcal{L}_v S \otimes T + S \otimes \mathcal{L}_v T$$

pf: pull back commutes with contraction and tensor

Dfn 4.8: Lie Bracket of vector fields: $[v, w] = \mathcal{L}_v w = -\mathcal{L}_w v$

acting on $f \in C^\infty(X)$, $[X, Y](f) = X(Y(f)) - Y(X(f))$

Lem 4.9: If $F: X \to Y$ is a diffeo, then $F^*(\mathcal{L}_v T) = \mathcal{L}_{F^* v}(F^* T)$

Dfn 4.10: $\alpha \in \Omega^r$, $v \in \mathfrak{X}$, then $\iota_v \alpha$ is α *contracted with* v:

$$v^{a_1} \alpha_{a_1 \dots a_r}$$

Lem 4.11 (Cartan's magic formula)

$$\mathcal{L}_v \alpha = d(\iota_v \alpha) + \iota_v (d\alpha)$$

5. Submanifolds, Foliations and Frobenius Integrability

Dfn 5.1: F is an immersion / submersion / local diffeo at p if DF is injective / surjective / an iso at p.

Regular point p = F submersion at p

Regular value q = $F^{-1}(q)$ contains only regular points. otherwise critical.

Lem 5.2: D_pF an iso, then $\exists$ open nhoods of p, $F(p)$ s.t $F|_U : U \to V$ a diffeo

Prop 5.4: $F: X \to Y$ an immersion at p, $x^1, \dots, x^n$ coords at p, then $\exists$ coords $y^1, \dots, y^n$ of Y s.t $y \circ F = (x^1, \dots, x^n, 0, \dots, 0)$

if submersion, then $\exists$ coords about p s.t $y \circ F = (x^1, \dots, x^m)$.

Dfn 5.5: Codim-k submanifold of X is a subset $Z \subseteq X$ s.t. $\forall p \in Z$, $\exists$ local coords $x_1, \dots, x_n$ about p s.t. Z is given by $x_1 = \dots = x_k = 0$.

Dfn 5.7: Smooth immersion that's homeo onto its image is an embedding.

Prop 5.10: If $F: X \to Y$ is smooth, and $q \in Y$ a regular value, then $F^{-1}(q)$ is a submanifold of X of codimension = $\dim(Y)$

Useful: $DF_p(x) = \frac{d}{dt} F(p + tx)\Big|_{t=0}$ from $\frac{d}{dt} f(t) = \frac{d}{dh}\Big|_{h=0} f(t+h)$

Thm 5.12 (Sard's Thm): for any smooth map $F: X \to Y$, {critical points} has measure 0 in Y $\to$ regular values are dense in Y

↳ ⚠ says nothing about regular points.

↳ $\dim X < \dim Y$, not a sub., so no reg. points. Reg values = $Y \setminus F(X)$.

Dfn 5.14: Submanifolds $Y, Z \subset X$ are transverse if $\forall p \in Y \cap Z$ $T_pY + T_pZ = T_pX$

Write $Y \pitchfork Z$

Prop 5.15: If $Y \pitchfork Z$, $\operatorname{codim}(Y) = k$, $\operatorname{codim}(Z) = \ell$, then $Y \cap Z$ is a submanifold, $\operatorname{codim}(Y \cap Z) = k + \ell$.

Dfn 5.16: A k-plane distⁿ D on X is a rank k vector subbundle of TX

↳ any k-plane distⁿ is locally the kernel of $n-k$ fibrewise lin. indep. 1-forms.

$= \operatorname{Ker}(\alpha_1, \dots, \alpha_{n-k})$

Y immersed curve ($\dot{Y}(t) \neq 0$ $\forall t$), ask $\dot{Y}(t) \in D_{Y(t)}$ $\forall t$. System of ODEs: $\alpha_i(\dot{Y}(t)) = 0$

$k=1$: $\exists!$ solution curve Y given by choosing some $Y(0)$ basepoint.

X locally $\simeq$ (c: = Disk transverse to D) $\times$ solution curves

$x^1, y^1, \dots, y^{n-1}$ coords for X, x^1 along solution curves, $y^1, \dots, y^{n-1}$ conserved quantities.

Solutions are given by y^i = constant.

$k > 1$: undetermined. Can hope that locally $\exists$ $n-k$ conserved quantities $y^1, \dots, y^{n-k}$ s.t

Solutions are level sets of the y^i.

Dfn 5.18: an ODE system of $n-k$ ODE's is called integrable if $\exists$ $n-k$ conserved (locally) quantities $y^1, \dots, y^{n-k}$ s.t Solutions are level sets of the y^i.

Dfn 5.19: An atlas $\{\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha\}$ is k-foliated if trans. funcs $\varphi_\beta \circ \varphi_\alpha^{-1}$ locally have the form $(x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}) \longmapsto (\xi(x,y), \eta(y))$

Two foliated atlases are equivalent if their union is also foliated. A foliation on X is an equivalence class of foliated atlases. Write foliated coordinates $x^1, \dots, x^k, y^1, \dots, y^{n-k}$

↳ X has slices that locally look like $\mathbb{R}^k \times$ {pts}.

Given a k-foliation of X, $\exists$ induced k-plane distⁿ $D = \langle \partial_{x^1}, \dots, \partial_{x^k} \rangle$ where x^i are the coordinates from the foliated atlas. These are the tangent spaces to the slices.

Thm 5.21 (Frobenius Integrability) A k-plane distⁿ D arises from a foliation in this way $\Leftrightarrow$ D is closed under $[\cdot,\cdot]$ (D is involutive)

Dfn 5.22: Such a distⁿ is called integrable.

Thm 5.24 (Frobenius integrability, alternate version) A k-plane distⁿ D arises from a k-foliation iff the annihilator

$$I(D) = \bigoplus_{r=0}^{n} \{ \alpha \in \Omega^r(X) : \forall p, \ v_1, \dots, v_r \in D_p, \ \alpha_p(v_1, \dots, v_r) = 0 \}$$

of D is closed under the exterior derivative d

6. Lie Groups and Lie Algebras

Dfn 6.1: Lie group = smooth manifold w/ smooth maps multiplication and inversion:

$\times : G \times G \to G$, $\quad {}^{-1} : G \to G$

Dfn 6.3: Lie subgroup = embedded submanifold that's also a subgroup

Dfn 6.5: $g \in G$, have maps:
$$\left.\begin{array}{l} L_g : h \mapsto gh \\ R_g : h \mapsto hg \\ C_g : h \mapsto ghg^{-1} \end{array}\right\} \text{diffeomorphisms}$$

Left invariant tensor: $(L_g)_* T = T$ (i.e. $(L_g)_* T_h = T_{gh}$) $\forall g$.

Rem: $(L_{g^{-1}})^* = (L_g)_*$.

Lem 6.7: Correspondence
$$\left\{\begin{array}{l}\text{left-invar.} \\ (p,q)\text{-tensors}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}(p,q)\text{-tensors} \\ \text{at } h\end{array}\right\}$$

Based on: $T_g = (L_{gh^{-1}})_* T_h$

Dfn 6.10: Lie Algebra $\mathfrak{g}$ is $T_e G$

Prop 6.12: Define bracket $\xi, \eta \in \mathfrak{g}$, $[\xi,\eta] = \ell_{[\xi,\eta]}(e)$, where

$\ell_{[\xi,\eta]}$ is left invar. v.f given by $[\ell_\xi, \ell_\eta]$.

Prop. 6.13: $\forall\ \xi \in \mathfrak{g}$, ℓ_ξ is complete

↳ solve $\dot\gamma(t) = \ell_\xi(\gamma(t))$, $\gamma(0) = e$ + note $\gamma(t+s) = \gamma(t)\gamma(s)$ for small s,t.

Then set $\Phi(t,g) = g\,\gamma(t)$.

Dfn 6.14: $\exp : \mathfrak{g} \to G$; $\exp(\xi) = \Phi_\xi^1(e)$

Lem 6.16: exp is smooth

Exm 6.17: $G = GL(n,\mathbb{R})$. Then $\mathfrak{g} \cong M_n(\mathbb{R})$, and $\exp : \mathfrak{g} \to G$ is
$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

Lem 6.18: $\xi, \eta \in \mathfrak{g}$, have $[\xi,\eta] = \frac{d}{dt}\Big|_{t=0} \left(C_{\exp(t\xi)}\right)_* \eta$

ES3 Q4: V, W v.f. on X, local flows Φ, Ψ. Small t and u, $\Phi^{-t} \circ \Psi^u \circ \Phi^t$

is local time-u flow of $(\Phi^t)^* W$. If $[V,W] = 0$, then Φ^t, Φ^u commute

Cor 6.20: If $\xi, \eta \in \mathfrak{g}$ satisfy $[\xi,\eta] = 0$, then $\exp(\xi + \eta) = \exp(\xi)\exp(\eta)$.

In particular, $\exp(\xi)$ and $\exp(\eta)$ commute.

Dfn 6.21: (left) action $\sigma : G \times X \to X$ smooth if map σ smooth

Dfn 6.23: rep$^{\text{n}}$: smooth action of G on a v.s. by linear maps: $\rho : G \to GL(V)$.

Exm 6.24: adjoint rep$^{\text{n}}$: action of G on $\mathfrak{g}$ by conjugation:
$$Ad_g(\xi) := (C_g)_* \xi.$$

Dfn 6.23: infinitesimal action of $\xi \in \mathfrak{g}$ on $x \in X$:
$$\xi \cdot x := D_{(e,x)}\sigma(\xi, 0) = (\exp(t\xi)x)^\cdot(0) \in T_x X$$

Infinit. adjoint action: $(Ad_{\exp(t\xi)}\eta)^\cdot(0) = [\xi,\eta]$ (see exm 6.24).

Thm 6.27: G action free + proper, then X/G a top. mani of $\dim = \dim(X) - \dim(G)$.

$\exists!$ smooth structure that makes $\pi : X \to X/G$ a submersion

Dfn 6.28: proper action: if $\sigma : G \times X \to X$ proper map. Equiv to: Say $(x_i), (g_i x_i)$

are two convergent sequences. Then σ proper if (g_i) has a convergent subsequence.

Dfn 6.29: Homogeneous space for G is a space X w/ transitive G-action

principal hom. space = G torsor := X w/ a free and transitive G-action.

7. Principal bundles and Connections.

7.1 Connections by hand

$\pi : E \to B$, trivialisations Φ_α, section s gives local $\mathbb{R}^k$ valued function v_α.

Dfn 7.1: Connection $\mathcal{A}$ on E is a $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form A_α on each coord patch

s.t. on overlaps $\quad A_\alpha = g_{\beta\alpha}^{-1} dg_{\beta\alpha} + g_{\beta\alpha}^{-1} A_\beta\, g_{\beta\alpha}$

Covariant derivative of s along $\mathcal{A}$, $d^{\mathcal{A}} s$, locally given by $dv_\alpha + A_\alpha v_\alpha$.

s horizontal / covariantly constant: $= d^{\mathcal{A}} s = 0$.

Dfn 7.3: Frame bundle: $F(E) = \bigsqcup_\alpha U_\alpha \times F(\mathbb{R}^k) \big/ (p, v_1, \dots, v_k) \sim (p, g_{\beta\alpha}(p) v_1, \dots, g_{\beta\alpha}(p) v_k)$

Natural right $GL(k,\mathbb{R})$-action: $g \cdot (p, v_1, \dots, v_k) \mapsto (p, v_1 g, \dots, v_k g)$

sections of $F(E) \Leftrightarrow$ trivialisations of E over U.

Connection on E $\to$ Connection on $F(E)$:

- Φ_α trivialisation of E $\to$ f_α section on $F(E)$.

  diffeo $\Phi_\alpha^F : \pi_F^{-1}(U_\alpha) \to U_\alpha \times GL(k,\mathbb{R})$; $f_\alpha(b) g \mapsto (b, g)$
- Define 1-forms on $U_\alpha \times GL(k,\mathbb{R})$ using A_α connections of E (which are defined on U_α)

  via: $(v \in T_b U_\alpha,\ g \cdot \xi \in T_g GL(k,\mathbb{R})) \mapsto Ad_{g^{-1}} A_\alpha(v) + \xi$
- pull back via Φ_α^F to $\mathfrak{gl}(k,\mathbb{R})$ valued 1-form on $\pi_F^{-1}(U_\alpha)$.

Prop 7.4: Local constructions agree on overlaps, and give $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form $\mathcal{A}$

on $F(E)$ satisfying:

$\mathcal{A}_p(p \cdot \xi) = \xi \quad \forall p \in F(E),\ \xi \in \mathfrak{gl}(k,\mathbb{R})$

$R_g^* \mathcal{A} = Ad_{g^{-1}} \mathcal{A} \quad \forall g \in GL(k,\mathbb{R})$

Conversely: $\mathcal{A}$ on $F(E)$ satisfying two conditions gives a connection on E via $A_\alpha = f_\alpha^* \mathcal{A}$.

7.2. Principal Bundles

Dfn 7.6: (principal) G-bundle is mani P w/

- smooth surjection $\pi : P \to B$
- $\{U_\alpha\}$ covering B + $\forall \alpha$ a diffeomorphism

$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ (trivialisations)

s.t (1) $pr_1 \circ \Phi_\alpha = \pi$,

(2) $\Phi_\beta \circ \Phi_\alpha^{-1}(b,g) = (b, g_{\beta\alpha}(b)g)$, $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$

Right G-action: $(b,g) \mapsto (b, gh)$.

sections + trivialisations correspondence:

$$\left\{ \begin{array}{l} \Phi \to s : \quad s(b) := \Phi^{-1}(b,e) \\ s \to \Phi : \quad \Phi^{-1}(b,g) := s(b)g \end{array} \right\}$$

Right action free + proper $\to$ $P/G \simeq B$

Dfn 7.9: $P \xrightarrow{\pi} B$, $\rho : G \to GL(V)$ a repⁿ, then *associated vector bundle*

$$P \times_G V := {}^{P \times V}/_{(pg, v) \sim (p, \rho(g)v)}$$

transition functions: $\rho(g_{\beta\alpha})$.

Exm 7.10: Adjoint bundle: $\rho : G \to GL(\mathfrak{g})$ adjoint repⁿ: $\xi \mapsto \overbrace{(C_g)_* \xi}^{(Ad_g \xi)}$,

then ass. v.b. is ad(P). If P = F(E), then ad(P) = End(E) = $E^\vee \otimes E$.

7.3. Connections

$\pi : P \to B$ a G-bundle

Dfn 7.11: A *connection* on P is a $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on P satisfying

- $\mathcal{A}_p(p \cdot \xi) = \xi$,
- $(R_g)^* \mathcal{A} = Ad_{g^{-1}} \mathcal{A}$.

Φ_α triv. of P, → ass. section s_α, then *local connection 1-forms* are

$$A_\alpha := s_\alpha^* \mathcal{A}.$$

Lem 7.12: $A_\alpha = g_{\beta\alpha}^{-1} dg_{\beta\alpha} \; Ad_{g_{\beta\alpha}^{-1}} A_\beta$

Prop 7.13: every principal bundle admits a connection:

↳ P w/ triv. Φ_α, U_α, define $\mathcal{A}_\alpha$ on $\pi^{-1}(U_\alpha)$ by $A_\alpha = 0$.

↳ $\{\rho_\alpha\}$ ubord to U_α, and then set $\mathcal{A} = \sum_\alpha (\rho_\alpha \circ \pi)\mathcal{A}_\alpha$.

Dfn 7.15: $p \in P$, *vertical subspace* $T_p^v P = \ker D_p \pi = p \cdot \mathfrak{g}$

horizontal subspace is any complementary subspace

Rem: $H := \ker \mathcal{A}$ is a horizontal distⁿ on P

↳ $\mathcal{A}$ right equivariant ⇒ H right invariant

↳ given a right invariant hor. distⁿ H, ∃! connection $\mathcal{A}$ on P s.t $H = \ker \mathcal{A}$.

Section s of P is *horizontal* iff its tangent to H: $s^* \mathcal{A} = 0$.

7.4. Curvature

Dfn 7.17: $\mathcal{A}$ is *flat* if horizontal distⁿ $H = \ker \mathcal{A}$ is integrable

Prop 7.18: TFAE:

(i) $\mathcal{A}$ is flat

(ii) P is foliated by local horizontal sections

(iii) P has a horizontal section locally over each point in B

(iv) P can be covered by trivialisations Φ_α s.t all A_α are 0.

Dfn: *curvature* $\mathcal{F}$ of $\mathcal{A}$ is the $\mathfrak{g}$-valued 2-form:

$$d\mathcal{A} + \tfrac{1}{2}[\mathcal{A} \wedge \mathcal{A}]$$

where: two $\mathfrak{g}$-valued 1-forms σ, τ, $[\sigma \wedge \tau](X_1, X_2) = [\sigma(X_1), \tau(X_2)] - [\tau(X_1), \sigma(X_2)]$.

Thm 7.20: $\mathcal{A}$ flat $\iff$ $\mathcal{F} = 0$

Prop 7.21: s_α section corr. to Φ_α, write F_α for $s_\alpha^* \mathcal{F}$. Then F_α is a $\mathfrak{g}$-valued 2-form on U_α. These local expressions glue together to give an ad(P)-valued 2-form on B: $s_\beta = s_\alpha g_{\beta\alpha}^{-1}$ and w/s $F_\beta = Ad_{g_{\beta\alpha}(b)} F_\alpha$

7.5. Algebraic Structures

Given a connection $\mathcal{A}$ on a G-bundle $P \to B$ and a repⁿ $\rho : G \to GL(V)$, there's an induced connection on the associated vector bundle $E = P \times_G V$, given by local connection 1-forms $D\rho(A_\alpha)$

Can extend covariant derivative $d^{\mathcal{A}}$ to an exterior covariant derivative using the Leibniz rule: an E-valued p-form σ can locally be written as a sum of expressions $s \otimes \alpha$ where s is a section of E and α is a p-form. Then define

$$d^{\mathcal{A}}(s \otimes \alpha) = (d^{\mathcal{A}} s) \wedge \alpha + s \otimes d\alpha.$$

Prop 7.24: 2nd Bianchi identity: $d^{\mathcal{A}} F = 0$

8. Riemannian Geometry:

Dfn 8.1: inner product g on E is a section of $(E^\vee)^{\otimes 2}$ which is fibrewise symmetric and positive definite

Riemannian metric: inner product on TX

Lem 8.2: Every v.b. $E \to B$ admits an inner product.

Dfn 8.3: *Riemannian manifold*: (X, g) is a manifold equipped with a Riem. metric.

Dfn 8.4: $\mathcal{A}$ on E is *compatible* with g if g is covariantly constant wrt. induced connection on $(E^\vee)^{\otimes 2}$.

Important Examples:

$$S^n : \quad U_\pm = S^n \setminus \{(0, \dots, 0, \pm 1)\}$$

$$\varphi_\pm = \frac{1}{1 \mp y_{n+1}} (y_1, \dots, y_n)$$

$$\varphi_\pm^{-1}(\underline{u}) = \left(\frac{2\underline{u}}{\|\underline{u}\|^2 + 1}, \pm \frac{\|u\|^2 - 1}{\|u\|^2 + 1} \right)$$